MATH 2850: 5.7 - VARIATION OF PARAMETERS

SETUP: Suppose $y_c = c_1y_1 + c_2y_2$ is the solution to $y'' + p(x)y' + q(x)y = 0$.

We seek a particular solution to $y'' + p(x)y' + q(x)y = f(x)$ of the form $y_p = u_1y_1 + u_2y_2$.

Substituting $y_p = u_1y_1 + u_2y_2$ into $y'' + p(x)y' + q(x)y = f(x)$ gets messy ...

EXAMPLE: Show that making the simplifying assumption: $u_1'y_1 + u_2'y_2 = 0$ results in $u_1'y_1' + u_2'y_2' = f(x)$.

In other words, we get a system of first order ODEs: $\left\{ \begin{array}{rcl} u_1'y_1 + u_2'y_2 & = & 0 \\ u_1'y_1' + u_2'y_2' & = & f(x) \end{array} \right.$.

EXAMPLE: Use Cramer's Rule to solve the above system for u_1' and u_2' and show the answers are:

$$u_1' = -\frac{y_2 f(x)}{W(x)}, \qquad u_2' = \frac{y_1 f(x)}{W(x)}$$

Where $W(x)$ is the Wronskian of y_1 and y_2, which we know is never zero (why?)

Hence, y_p can be given by: $y_p = -y_1 \displaystyle\int \frac{y_2 f(x)}{W(x)}\, dx + y_2 \int \frac{y_1 f(x)}{W(x)}\, dx$

EXAMPLE: Solve: $y'' - 10y' + 25y = \dfrac{2e^{5x}}{x^2+1}$

Ans: $y = c_1 e^{5x} + c_2 x e^{5x} + 2xe^{5x}\tan^{-1}(x) - e^{5x}\ln\left(x^2+1\right)$

EXAMPLE: Solve: $\cos(2x)y'' + 4\cos(2x)y - 4\sin(2x) = 0$, $y(0) = 3$, $y'(0) = -6$

NOTE: Don't forget to put the DE In the form: $y'' + p(x)y' + q(x)y = f(x)$ first!

Ans: $y = 3\cos(2x) - 2\sin(2x) - \cos(2x)\ln|\sec(2x) + \tan(2x)|$

EXAMPLE: Solve $x^2y'' + 2xy' - 2y = 9x$ for $x > 0$ given that $y_c = c_1x + c_2x^{-2}$.

NOTE: Don't forget to put the DE In the form: $y'' + p(x)y' + q(x)y = f(x)$ first!

Ans: $y = c_1x + c_2x^{-2} + 3x\ln(x),\ x > 0$

EXAMPLE: Solve $y'' - y = \dfrac{2}{\sinh(x)}$ for $x > 0$.

RECALL: $\cosh^2(x) - \sinh^2(x) = 1$, $D_x[\cosh(x)] = \sinh(x)$, and $D_x[\sinh(x)] = \cosh(x)$,

Ans: $y = c_1 \sinh(x) + c_2 \cosh(x) + 2\sinh(x)\ln(\sinh(x)) - 2x\cosh(x)$.

HOMEWORK: Pg. 262: 1 - 33 e.o.o.